FOLIA 196

Annales Universitatis Paedagogicae Cracoviensis

Studia ad Didacticam Mathematicae Pertinentia VII (2015)

*Jan Górowski, Adam Łomnicki*

# Quasi-arithmetic means*

**Abstract.** We present a list of geometric problems with solutions that lead to known or less known means. We also prove, by elementary means, some property for so-called quasi-arithmetic means. We use the proved result to justify some inequalities between the means.

## 1. Introduction

Let $J \subset \mathbb{R}$ denote the open interval or respectively closed or half-closed. The sets $\mathbb{R}$, $\mathbb{R}_+ := (0, +\infty)$ and $\mathbb{R}_+ \cup \{0\}$ will be also considered as intervals.

One of the most general definition of a mean is the following

Definition 1

Every function $d\colon J \times J \to J$ satisfying

(i) $\forall a, b \in J \quad \min\{a, b\} \le d(a, b) \le \max\{a, b\}$,

(ii) $d$ is a increasing function with respect to each variable

is called a mean.

In (Aczél, 1948) and (Kitagawa, 1934) it was proved that under some additional conditions on $d$ there exists a strictly monotone function $g$ defined on $J$ such that

$$d(a, b) = g^{-1}(pg(a) + qg(b)), \qquad a, b \in J,$$

for some $p, q \in (0, 1)$ such that $p+q = 1$. Such means will be called quasi-arithmetic means.

For the purposes of this paper we modify Definition 1.

Definition 2

Let $T = \{(x, y) \in J \times J : \ x \ge y\}$. Every function $d\colon T \to J$ satisfying conditions (i) and (ii) of Definition 1 is said to be a mean.

*Średnie quasi-arytmetyczne

2010 Mathematical Subject Classification: Primary: 26E60, 26D07

Key words and phrases: Quasi-arithmetic means, inequalities involving means, extended mean values, means in geometry

In the sequel by a mean we understand a function in a sense of Definition 2.

There is a wide literature on means, some information may be found in (Aczél, 1948; Aczel, Dhombres, 1989; Galwani, 1927; Głazowska, Jarczyk, Matkowski, 2002; Górowski, Łomnicki, 2010; Kitagawa, 1934; Kołgomorov, 1930; Leach, Sholander, 1978; Leach, Sholander, 1983; Witkowski, 2009).

## 2. Geometric problems leading to means

Let us assume that a quadrangle $ABCD$ (see Fig. 1.) is a trapezium such that $AB \parallel CD$, $|AB| = a$ $|DC| = b$ and according to Fig. 1. $EF \parallel AB$, $|EF| = d$ and $DA' \parallel CB$. Denoting $\lambda = \frac{|AE|}{|ED|}$ we express $d$ as a function of $\lambda$. By The Intercept

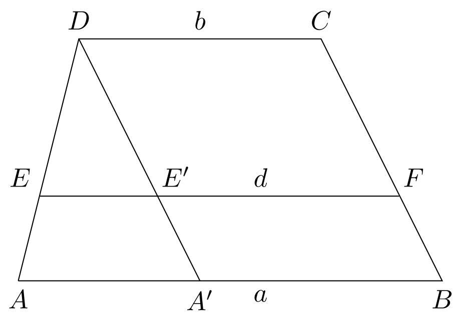


Fig. 1.

Theorem we get

$$\frac{|AA'|}{|EE'|} = \frac{|AD|}{|ED|} = \frac{|AE| + |ED|}{|ED|} = \lambda + 1.$$

Hence

$$|EE'| = \frac{|AA'|}{\lambda + 1} = \frac{a - b}{\lambda + 1} \quad \text{and} \quad d = \frac{a - b}{\lambda + 1} + b = \frac{a + \lambda b}{\lambda + 1}$$

and thus

$$d = \frac{a + \lambda b}{\lambda + 1}. \tag{1}$$

Now we formulate some geometric problems leading to means. Notice the well known problems P1-P4.

Problem 1
Find the length $d$ of the segment $EF$ in the $ABCD$ (see Fig 1.) if

**P1.** $E$, $F$ are the midpoints of the segments $AD$ and $BC$, respectively;

**P2.** the diagonals $AC$ and $BD$ and the segment $EF$ intersect at a point;

**P3.** the trapezes $ABFE$ and $EFCD$ are similar;

**P4.** the areas of the trapezes $ABFE$ and $EFCD$ are equal;

**P5.** the volumes of the solids of revolution obtained by rotating $ABFE$ and $EFCD$ around the line $EF$ are equal;

**P6.** the volumes of the solids of revolution obtained by rotating $ABFE$ around the line $AB$ and $EFCD$ around the line $DC$ are equal;

It is easy to see that the solution of P1 is $d = \frac{a+b}{2}$.

Denote by $S$ the intersection point of the diagonals $AC$ and $BD$ and the segment $EF$ (problem P2). Then by Intersection Theorem we get

$$\lambda = \frac{|AE|}{|ED|} = \frac{|AS|}{|SC|} = \frac{|AB|}{|DC|} = \frac{a}{b}.$$

thus

$$d = \frac{2ab}{a+b}.$$

For the problem P3 notice that since the trapezes $ABFE$ and $EFCD$ are similar we obtain

$$\frac{|AE|}{|ED|} = \frac{|AB|}{|EF|} = \frac{|EF|}{|DC|},$$

hence $d^2 = ab$, and $d = \sqrt{ab}$.

To solve P4 denote by $h_1$, $h_2$ the altitudes of the trapezes $ABFE$ and $EFCD$, respectively. Let $P$ denotes the area of the trapezium $ABFE$ (also trapezium $EFCD$). Then

$$\lambda = \frac{|AE|}{|ED|} = \frac{h_1}{h_2} = \frac{P}{a+d} \cdot \frac{d+b}{P} = \frac{d+b}{a+d}.$$

This and (1) give

$$d = \sqrt{\frac{a^2+b^2}{2}}.$$

Now (problem P5) let $h_1$, $h_2$ be defined as above, then

$$\lambda^2 = \frac{h_1^2}{h_2^2} = \frac{\pi h_1^2}{\pi h_2^2}. \tag{2}$$

On the other hand, the volumes of the solids of revolution obtained by rotating $ABFE$ and $EFCD$ around the line $EF$ are equal

$$\pi h_1^2 d + \frac{2}{3}\pi h_1^2 (a-d), \quad \pi h_2^2 b + \frac{1}{3}\pi h_2^2 (d-b), \tag{3}$$

respectively. From (2) and (3) we get

$$\lambda^2 = \frac{\pi h_1^2 (d + \frac{2}{3}(a-d))}{\pi h_2^2 (b + \frac{1}{3}(d-b))} \cdot \frac{b + \frac{1}{3}(d-b)}{d + \frac{2}{3}(a-d)} = \frac{b + \frac{1}{3}(d-b)}{d + \frac{2}{3}(a-d)} = \frac{d+2b}{d+2a},$$

which by (1) yields

$$d = \frac{2(a^2+ab+b^2)}{3(a+b)} = \frac{\frac{a^3-b^3}{3}}{\frac{a^2-b^2}{2}}. \tag{4}$$

Finally, for the solution of P6 observe that

$$V_1 = \pi h_1^2 d + \frac{1}{3}\pi h_1^2(a-d) \quad \text{and} \quad V_2 = \pi h_2^2 b + \frac{2}{3}\pi h_2^2(d-b),$$

where $h_1$, $h_2$ denote the altitudes of the trapezes $ABFE$ and $EFCD$, resp., and $V_1$, $V_2$ are the volumes of the solids of revolution obtained by rotating $ABFE$ around the line $AB$ and $EFCD$ around the line $DC$, resp. Similarly as above we get

$$\lambda^2 = \frac{b+2d}{a+2d},$$

thus

$$d = \sqrt{\frac{a^2+ab+b^2}{3}} = \sqrt{\frac{a^3-b^3}{3(a-b)}}. \tag{5}$$

Observe that (4) is one of the means introduced by Leach and Sholander in (Leach, Sholander, 1978), and (5) is a Stolarsky's mean from (Kołgomorov, 1930).

Problem 2

Consider three pairwise homothetic squares with side length $a > d > b$, see Fig 2. Find $d$ in terms of $a$ and $b$.

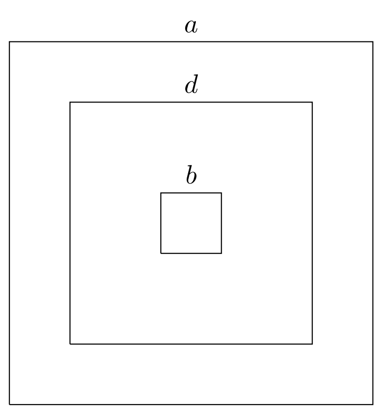


Fig. 2.

Let $\lambda = \frac{a^2-d^2}{d^2-b^2}$. Then

$$d = \sqrt{\frac{a^2+\lambda b^2}{1+\lambda}}.$$

This yields the following relationships:

$$
\begin{aligned}
d &= \sqrt{\frac{a^2+b^2}{2}} && \text{for } \lambda = 1,\\
d &= \sqrt{ab} && \text{for } \lambda = \frac{a}{b},\\
d &= \frac{a+b}{2} && \text{for } \lambda = \frac{3a+b}{a+3b},\\
d &= \frac{2ab}{a+b} && \text{for } \lambda = \frac{a^2(a+3b)}{b^2(3a+b)},\\
d &= \sqrt{\frac{a^3+b^3}{a+b}} && \text{for } \lambda = \frac{b}{a}.
\end{aligned}
$$

## 3. Quasi-arithmetic means

Considerations from the previous section imply that, under some assumptions on a function $g\colon J \to \mathbb{R}$ and $\lambda$, it is worth to consider the function $d_g^\lambda$ defined on $T = \{(x,y) \in J \times J : \ x \geq y\}$ and given by

$$d_g^\lambda(a,b) = g^{-1}\left(\frac{g(a)+\lambda g(b)}{1+\lambda}\right), \qquad (a,b) \in T. \tag{6}$$

we prove now the following result.

Theorem 1
*If $g\colon J \to \mathbb{R}$ is a strictly monotonic function, continuous on $J$ and $\lambda$ is a non-negative real number, then $d_g^\lambda$ given by* (6) *is a mean (in a sense of Definition 2).*

*Proof.* It is easy to see that $d_g^\lambda$ is well defined. Indeed, if for some $(a,b) \in T$ and some $\lambda \in [0,+\infty)$ we had

$$\frac{g(a)+\lambda g(b)}{1+\lambda} - g(x) \neq 0 \quad \text{for every } x \in J,$$

then since $g$ is continuous,

$$\frac{g(a)+\lambda g(b)}{1+\lambda} - g(x) > 0 \quad \text{for every } x \in J \tag{7}$$

or

$$\frac{g(a)+\lambda g(b)}{1+\lambda} - g(x) < 0 \quad \text{for every } x \in J. \tag{8}$$

From (7) we obtain the following system of inequalities

$$\frac{g(a)+\lambda g(b)}{1+\lambda} - \frac{(1+\lambda)g(a)}{1+\lambda} > 0 \quad \text{and} \quad \frac{g(a)+\lambda g(b)}{1+\lambda} - \frac{(1+\lambda)g(b)}{1+\lambda} > 0,$$

which leads to a contradiction. The similar argument can be applied to (8).

The task is now to show that

$$\min\{a,b\} \le d_g^\lambda(a,b) \le \max\{a,b\}, \qquad (a,b) \in T. \tag{9}$$

Observe that if $g$ is a strictly increasing function, then so is $g^{-1}$ and (9) is equivalent to

$$g(b) \le \frac{g(a)+\lambda g(b)}{1+\lambda} \le g(a),$$

$$(1+\lambda)g(b) \le g(a)+\lambda g(b) \le (1+\lambda)g(a),$$

where the last inequality holds true. Similar argument applies to the case when $g$ is strictly decreasing.

Finally, we prove that $d_g^\lambda$ is an increasing function with respect to each variable. Fix $a \in J$ and suppose that $g$ is strictly increasing. Let $b_1, b_2 \in J$ be such that $b_1 > b_2$ and $a \ge b_1$, then $g(b_1) > g(b_2)$, $\lambda g(b_1) \ge \lambda g(b_2)$, $g(a)+\lambda g(b_1) \ge g(a)+\lambda g(b_2)$ and in a consequence $d_g^\lambda(a,b_1) \ge d_g^\lambda(a,b_2)$. Now fix $b \in J$ and assume that $a_1 > a_2 \ge b$ for arbitrary $a_1, a_2 \in J$. We have $g(a_1) > g(a_2)$, $g(a_1)+\lambda g(b) > g(a_2)+\lambda g(b)$ and $d_g^\lambda(a_1,b) \ge d_g^\lambda(a_2,b)$.

For a strictly decreasing $g$ the proof runs similarly.

Definition 3

Let $g$ satisfies the assumptions of Theorem 1. Every function defined by (6) will be called a mean generated by pair $(g,\lambda)$.

Theorem 2

*Let $d_g^\lambda$ be a mean on a set $T$ generated by pair $(g,\lambda)$. A function $\psi\colon [0,+\infty) \to \mathbb{R}^{T\setminus\{(a,a):a\in J\}}$ defined by*

$$\psi(\lambda) = \bar{d}_g^\lambda,$$

*where $\bar{d}_g^\lambda$ is a restriction of $d_g^\lambda$ to the set $T \setminus \{(a,a) : a \in J\}$ is strictly decreasing.*

*Proof.* Fix $a, b \in J$ such that $a > b$ and put

$$\phi(\lambda) := \frac{g(a)+\lambda g(b)}{1+\lambda}, \qquad \lambda \in [0,+\infty).$$

It follows that

$$\phi'(\lambda) := \frac{g(b)-g(a)}{(1+\lambda)^2},$$

thus $\phi$ is strictly increasing (resp. strictly decreasing) if $g$ is strictly decreasing (resp. strictly increasing). Hence for $\lambda_1 < \lambda_2$ we have $d_g^{\lambda_1}(a,b) > d_g^{\lambda_2}(a,b)$ and $\bar{d}_g^{\lambda_1} > \bar{d}_g^{\lambda_2}$, which completes the proof.

## 4. Means generated by the identity function

Suppose that $g = \mathrm{Id}_{\mathbb{R}_+}$, where $\mathrm{Id}_{\mathbb{R}_+}(x) = x$ for $x \in \mathbb{R}_+$, then (6) becomes

$$d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b) = \frac{a+\lambda b}{1+\lambda}, \qquad (a,b) \in \mathbb{R}_+ \times \mathbb{R}_+.$$

Some of the means of this kind appeared in Problem 1. (problems P1-P6).

Now using Theorem 2 we establish some inequalities between means generated by pair $(\mathrm{Id}_{\mathbb{R}_+}, \lambda)$. Fix $a, b \in \mathbb{R}_+$ such that $a > b$, then

$$\frac{a}{b} > \sqrt{\frac{a}{b}} > 1 > \frac{\sqrt{\frac{a^2+b^2}{2}} + b}{\sqrt{\frac{a^2+b^2}{2}} + a},$$

which yields the following relation between the harmonic, geometric, arithmetic and quadratic mean of $a$ and $b$,

$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}}.$$

Moreover, the means from problems P5 and P6 are greater than the arithmetic mean. Indeed, solving problem P5 we proved that for $a > b$,

$$\lambda^2 = \frac{2b + d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b)}{2a + d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b)}$$

which means that

$$\lambda < 1 \quad \text{and} \quad \frac{2}{3}\frac{a^3-b^3}{a^2-b^2} > \frac{a+b}{2}.$$

By a similar argument, from equality obtained in the solution of problem P6,

$$\lambda^2 = \frac{b + 2d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b)}{a + 2d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b)}$$

it follows that for $a > b$,

$$\sqrt{\frac{a^3-b^3}{3(a-b)}} > \frac{a+b}{2}.$$

To end this section let us remark that for arbitrary fixed $a, b \in \mathbb{R}_+$ such that $a > b$ we have

$$\left(\frac{a}{b}\right)^{\mu} > 1 \text{ for } \mu > 0 \quad \text{and} \quad \left(\frac{a}{b}\right)^{\mu} < 1 \text{ for } \mu < 0,$$

thus for $\lambda = \left(\frac{a}{b}\right)^{\mu}$,

$$d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b) = \frac{ab^{\mu} + ba^{\mu}}{a^{\mu} + b^{\mu}} < \frac{a+b}{2} \quad \text{for } \mu > 0$$

and

$$d^{\lambda}_{\mathrm{Id}_{\mathbb{R}_+}}(a,b) = \frac{ab^{\mu} + ba^{\mu}}{a^{\mu} + b^{\mu}} > \frac{a+b}{2} \quad \text{for } \mu < 0.$$

## 5. Some other generated means

In this section we consider means generated by pair $(g,\lambda)$, where $g\colon \mathbb{R}_+ \to \mathbb{R}$ is a power or a logarithmic function.

Let $g(x) = x^\nu$, $x \in \mathbb{R}_+$, $\nu \in \mathbb{R} \setminus \{0\}$. Then

$$d_g^\lambda(a,b) = \left(\frac{a^\nu + \lambda b^\nu}{1+\lambda}\right)^{\frac{1}{\nu}}, \quad (a,b) \in \{(x,y) : x \in \mathbb{R}_+,\ x \geq y\}.$$

By Theorem 2 it follows that for $a > b$ and $\mu > 0$ we have $\left(\frac{a}{b}\right)^\mu > 1$ and

$$d_g^\lambda(a,b) < d_g^1(a,b),$$

hence for $\lambda = \left(\frac{a}{b}\right)^\mu$,

$$\left(\frac{a^\nu b^\mu + a^\mu b^\nu}{a^\mu + b^\mu}\right)^{\frac{1}{\nu}} < \left(\frac{a^\nu + b^\nu}{2}\right)^{\frac{1}{\nu}}.$$

Similarly, for $\mu < 0$ we get

$$\left(\frac{a^\nu b^\mu + a^\mu b^\nu}{a^\mu + b^\mu}\right)^{\frac{1}{\nu}} > \left(\frac{a^\nu + b^\nu}{2}\right)^{\frac{1}{\nu}}.$$

Now suppose that $g(x) = \ln x$, $x \in \mathbb{R}_+$. We have

$$d_g^\lambda(a,b) = \exp\frac{\ln a + \lambda \ln b}{1+\lambda} = (ab^\lambda)^{\frac{1}{1+\lambda}}, \quad (a,b) \in \{(x,y) : x \in \mathbb{R}_+,\ x \geq y\}.$$

Setting again $\lambda = \left(\frac{a}{b}\right)^\mu$, $\mu \in \mathbb{R} \setminus \{0\}$ we get

$$d_g^\lambda(a,b) = a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}}$$

and from Theorem 2 the following inequalities

$$a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}} < \sqrt{ab} \quad \text{for } \mu > 0,$$

$$a^{\frac{b^\mu}{a^\mu + b^\mu}} b^{\frac{a^\mu}{a^\mu + b^\mu}} > \sqrt{ab} \quad \text{for } \mu < 0.$$

Notice that every strict inequality obtained by Theorem 2 for $(a,b) \in \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ :\ x \geq y\}$ if replaced by a its corresponding non-strict inequality holds true for $(a,b) \in \mathbb{R}_+ \times \mathbb{R}_+$.

## References

Aczél, J.: 1948, On mean values, *Bull. Amer. Math. Soc.* **54**(4), 392-400.

Aczel, J., Dhombres, J.: 1989, *Functional equation in several variables*, Vol. 31, Cambridge Univ. Press, Cambridge-New York-Rochelle-Melbourne-Sydney.

Galwani, L.: 1927, Dei limiti a cui tendono alcune media, *Boll. Un. Math. Ital.* **6**, 173-179.

Głazowska, D., Jarczyk, W., Matkowski, J.: 2002, Arithmetic mean as a linear combination of two quasi-arithmetic means, *Publ. Math. Debrecen* **61**, 455-467.

Górowski, J., Łomnicki, A.: 2010, O średnich, *Ann. Univ. Paed. Cracov. Studia ad Didacticam Math. Pertinentia* **3**, 55-66.

Kitagawa, T.: 1934, On some class of weighted means, *Proc. Phys.-Math. Soc. Japan* **16**(3rd series), 117-126.

Kołgomorov, A.: 1930, Sur la notion de la moyenne, *Alti Accad. Naz. Lincei* **12**(6), 388-391.

Leach, E., Sholander, M.: 1978, Extended mean values, *Amer. Math. Monthly* **85**, 84-90.

Leach, E., Sholander, M.: 1983, Extended mean values ii, *J. Math. Appl.* **92**, 207-223.

Witkowski, A.: 2009, Comparison theorem for two-parameter means, *Math. Inequal. Appl.* **12**, 11-20.

*Instytut Matematyki*
*Uniwersytet Pedagogiczny*
*ul. Podchorążych 2*
*PL-30-084 Kraków*
*e-mail* jangorowski@interia.pl
*e-mail* alomnicki@poczta.fm